PROJECTED WRITTEN NOTES FROM THE M408D LECTURE ON THURSDAY – April 25, 2024,

ON A REVIEW OF POWER SERIES REPRESENTATIONS (PSRs)

<u>Problem</u>: Consider the iterated integral

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy.$$

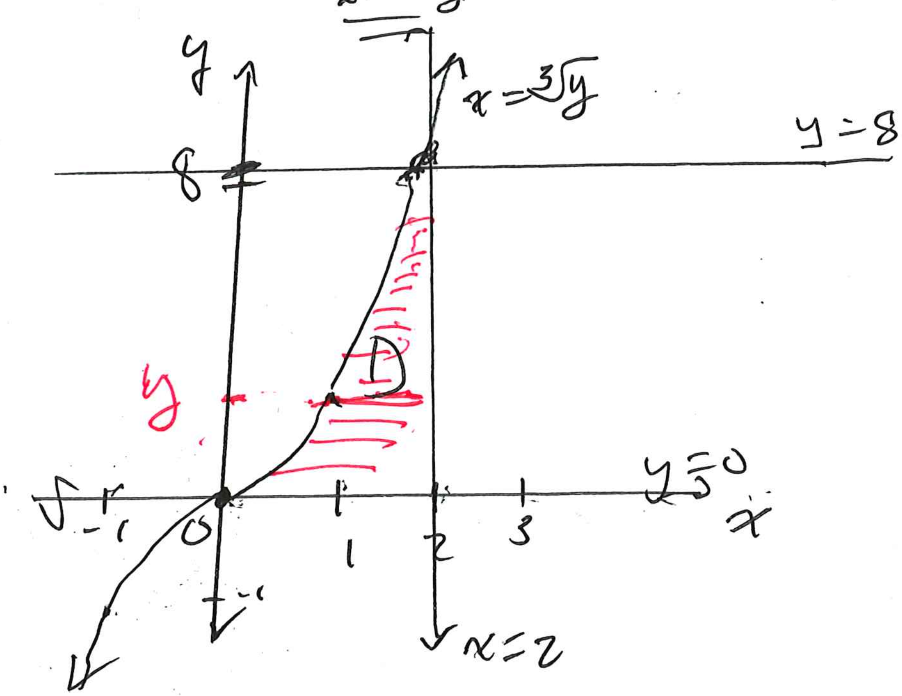
A) Determine the region D in the xy plane so that $\iint_D e^{x^4}\, dA = \int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy$

B) Evaluate $\iint_D e^{x^4}\, dA = \int_0^8 \int_{\sqrt[3]{y}}^2 \, dx\, dy$

<u>Sol'n</u>: $\int_0^8 \int_{\underline{\underline{\sqrt[3]{y}}}}^{2} e^{x^4}\, dx\, dy$ $\quad \underline{\underline{x=2}}$ $\quad \underline{\underline{x=\sqrt[3]{y}}}$

A type II Desc of D (dxdy)

D: (x, y) with
$0 \le y \le 8$
$\sqrt[3]{y} \le x \le 2$



$x = \sqrt[3]{y}$

$y = x^3$

B) Evaluate $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \iint_D e^{x^4}\,dA$

$$\int_0^8 \left(\int_{\sqrt[3]{y}}^2 e^{x^4}\,dx \right) dy = \int_0^8 \left(\begin{array}{c}\text{PROBLEM}\\ \text{TOO}\\ \text{HARD}\end{array} \right) dy$$

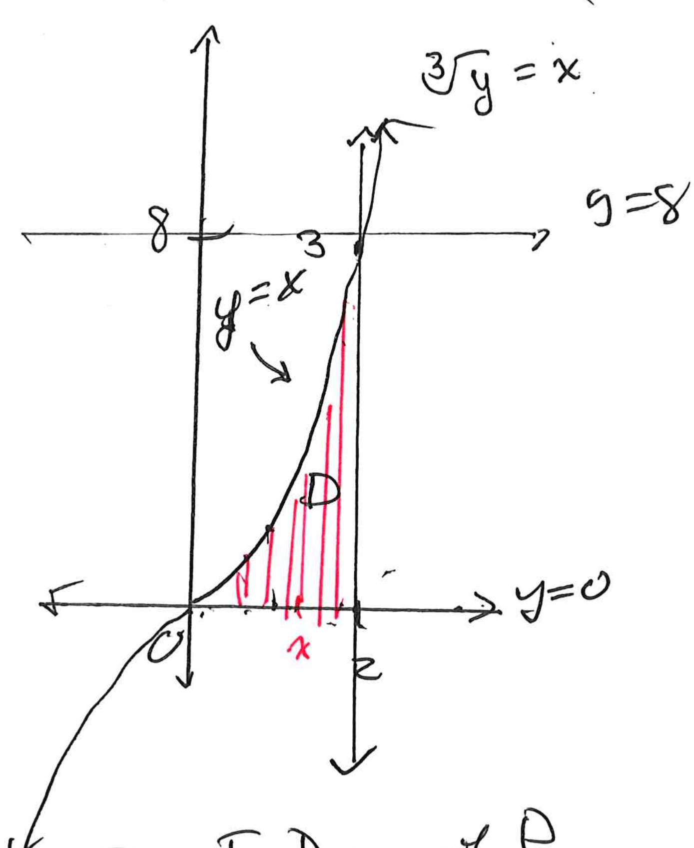
We must consider switching the order of integration from $dx\,dy$ to $dy\,dx$..

$$\iint_D e^{x^4}\,dA = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx$$

$$= \int_0^2 \left(\int_0^{x^3} e^{x^4}\,dy \right) dx$$

$$= \int_0^2 \left(y e^{x^4} \Big|_0^{x^3} \right) dx$$

$$= \int_0^2 \left(x^3 e^{x^4} \right) dx$$



Type I Desc of D

$0 \le x \le 2$

$0 \le y \le x^3$

$$= \frac{1}{4}\int_{0}^{16} e^{u}\,du$$

Let $u = x^4$

$du = 4x^3\,dx$

$x^3 dx = \frac{1}{4}du$

when $x=0$, $u=0$

when $x=2$, $u=16$

$$= \frac{1}{4}\left(e^{u}\Big|_{0}^{16} \right)$$

$$= \frac{1}{4}\left(e^{16} - e^{0}\right) = \frac{1}{4}\left(e^{16}-1\right)$$

$$\int_{0}^{2}\int_{0}^{x^3} e^{x^4}\,dy\,dx = \int_{0}^{8}\int_{\sqrt[3]{y}}^{2} e^{x^4}\,dxy = \frac{1}{4}\left(e^{16}-1\right)$$

Power Series Representations (PSRs)

Problem: Let $f(x) = \dfrac{1}{5+4x^2}$.

Find a PSR for $f(x) = \dfrac{1}{5+4x^2}$.

Sol'n; RECALL, $\dfrac{1}{1-a} = \sum_{n=0}^{\infty} a^n$ for $|a|<1$.

$$\frac{1}{5+4x^2} = \frac{1}{5(1+\frac{4}{5}x^2)} = \frac{1}{5}\left(\frac{1}{1+\frac{4}{5}x^2}\right)$$

$$= \frac{1}{5}\left(\frac{1}{1-(-\frac{4}{5}x^2)}\right) = \frac{1}{5}\sum_{n=0}^{\infty}\left(-\frac{4}{5}x^2\right)^n$$

$$\frac{1}{5+4x^2} = \frac{1}{5}\sum_{n=0}^{\infty}\left((-1)^n \frac{4^n}{5^n} x^{2n}\right).$$

$$\boxed{\begin{aligned}|{-\tfrac{4}{5}x^2}| &< 1\\ \tfrac{4}{5}|x|^2 &< 1\\ |x|^2 &< \tfrac{5}{4}\\ |x| &< \tfrac{\sqrt{5}}{2}\end{aligned}}$$

$$\boxed{\begin{gathered} f(x) = \frac{1}{5+4x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{5^{n+1}} x^{2n}. \\ \text{R.O.C.} = R = \frac{\sqrt{5}}{2} \end{gathered}}$$

Problem: Let $g(x) = \dfrac{3x^7}{5+4x^2}$

Find a PSR for
$$g(x) = \frac{3x^7}{5+4x^2}.$$

Sol'n:
$$g(x) = \frac{3x^7}{5+4x^2} = (3x^7)\left(\frac{1}{5+4x^2}\right).$$

From the previous problem:
$$\frac{1}{5+4x^2} = \sum_{n=0}^{\infty}\left((-1)^n \frac{4^n}{5^{n+1}} x^{2n}\right)$$
and $R = \dfrac{\sqrt{5}}{2}$.

$$g(x) = (3x^7)\left(\frac{1}{5+4x^2}\right)$$
$$= (3x^7)\sum_{n=0}^{\infty}\left((-1)^n \frac{4^n}{5^{n+1}} x^{2n}\right)$$

$$g(x) = \sum_{n=0}^{\infty}\left((-1)^n \frac{3\times 4^n}{5^{n+1}} x^{(2n+7)}\right),$$

$$R = \frac{\sqrt{5}}{2}.$$

Problem: Let $f(x) = \frac{1}{3+2x}$.

Find a PSR for $f'(x)$.

Sol'n: $f(x) = \frac{1}{3+2x} = (3+2x)^{-1}$

$$f'(x) = (-1)(3+2x)^{-2} \cdot 2$$

$$f'(x) = -\frac{2}{(3+2x)^2}$$

STOP

GO BACK TO $f(x)$

$f(x) = \frac{1}{3+2x}$, Get a PSR for

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

ROC

$$f'(x) = \sum_{n=0}^{\infty} c_n \, n \cdot x^{n-1}$$

SAME ROC

Continued on next page.

HOW TO WORK THIS PROBLEM:

① FIND A PSR for $f(x) = \frac{1}{3+2x}$

② $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with ROC R.

③ Differentiate the power series for $f(x)$ term-by-term to produce a PSR for $f'(x)$ with the SAME ROC.

④ $f'(x) = \sum_{n=0}^{\infty} n c_n x^{n-1}$

⑤ Get the sum with no initial terms equal to 0 and then get the sum in terms of x^n, not in terms of x^{n-1}.

$$f(x) = \frac{1}{3+2x} = \frac{1}{3(1+\frac{2}{3}x)} = \frac{1}{3}\left(\frac{1}{1-(-\frac{2}{3}x)}\right)$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \left(-\frac{2}{3}x\right)^n = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^n} x^n$$

$r = -\frac{2}{3}x$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^{n+1}} x^n.$$

$|r| < 1$

$|-\frac{2}{3}x| < 1$

ROC $R = \frac{3}{2}$

$\frac{2}{3}|x| < 1$

$|x| < \frac{3}{2}$

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^{n+1}} (n x^{n-1})$$

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n n \cdot \frac{2^n}{3^{n+1}} x^{n-1}$$

ROC $R = 3/2$.

FIRST DROP Initial Terms equal to 0

$$f'(x) = \sum_{n=1}^{\infty} (-1)^n n \frac{2^n}{3^{n+1}} x^{n-1}, \quad R = 3/2$$

Then Rewrite with x^n instead of x^{n-1}

To Change $n-1$ to n

$+1$ $+1$

CHANGE n to $n+1$

$$\boxed{f'(x) = \sum_{n=0}^{\infty} (-1)^{n+1} (n+1) \frac{2^{n+1}}{3^{n+2}} x^n \qquad R = 3/2}$$

$n+1 = 1$

$n = 0$